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The module category of the Iwahori-Hecke algebra in non-integral rank

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Introduction

Throughout this report, we fix a commutative ring $\Bbbk$ and a parameter $q \in \Bbbk$. For a natural number $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, we denote by $H_n = H_n(q)$ the *Iwahori–Hecke algebra* of type A_{n-1} generated by elements $T_1, T_2, \dots, T_{n-1}$ with defining relations

$$T_iT_j = T_jT_i \quad \text{if } |i-j| \geq 2, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad (T_i - q)(T_i + 1) = 0.$$

Let $\mathfrak{S}_n$ denotes the symmetric group of rank n. Then it is known that for each $w \in \mathfrak{S}_n$ with a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ the element $T_w := T_{i_1}T_{i_2}\cdots T_{i_r}$ is well-defined, and that the set $\{T_w \,|\, w \in \mathfrak{S}_n\}$ forms a basis of H_n. Hence it is considered as a q-analogue of the symmetric group algebra $\Bbbk\mathfrak{S}_n$. The Iwahori–Hecke algebra H_n is one of the most important algebra in representation theory. It first comes from a study of flag varieties over the finite fields, and also appears as an endomorphism algebra of a certain representation of the quantum general linear group via an analogue of the Schur–Weyl duality.

Now let H_n-$\mathcal{M}od$ denotes the (left) module category of H_n. In his recent work the author introduce a family of new categories $\underline{H}_t$-$\mathcal{M}od$ indexed by a parameter t which is not necessarily an integer, which "interpolates" ordinary module categories H_n-$\mathcal{M}od$ for $n \in \mathbb{N}$ in the following sense. First we introduce an index set $B_q(\Bbbk)$ which contains $\mathbb{N}$ as a subset. An element of $B_q(\Bbbk)$ is called a *q-binomial sequence* in $\Bbbk$. Now for a while we assume that $q \in \Bbbk$ is invertible for simplicity. Then for each q-binomial sequence $t \in B_q(\Bbbk)$, the category $\underline{H}_t$-$\mathcal{M}od$ is defined. We call an object in $\underline{H}_t$-$\mathcal{M}od$ a *fakemodule* over $\underline{H}_t$, though "the algebra $\underline{H}_t$" itself does not really exist. When $t = n \in \mathbb{N}$, there is an equipped functor

$$P\colon \underline{H}_n\text{-}\mathcal{M}od \to H_n\text{-}\mathcal{M}od$$

called the *realization functor*, which sends $\underline{H}_n$-fakemodules to ordinary H_n-modules. This realization functor is full and surjective, so the category H_n-$\mathcal{M}od$ can be identified with a quotient category of $\underline{H}_n$-$\mathcal{M}od$. The structure of the fakemodule category $\underline{H}_t$-$\mathcal{M}od$ captures a behavior of stable structures of usual H_n-$\mathcal{M}od$ for $n \gg 0$. It is sometimes simpler than the usual ones, since its hom-spaces almost do not depend on the choice of t. Based on this property, its super-version $\underline{H}_t^c$-$\mathcal{M}od$, which is the module category of the *Hecke–Clifford superalgebra* H_n^c in non-integral rank, is used by the author [**Mor14**] to determine the generalized cellular structure of the ordinal module category of H_n^c.

This work is a part of "representation theory in non-integral rank" developed by Deligne. In his study of tensor categories, he defined the representation category of linear algebraic groups GL_t [**DM82, Del90**], O_t and Sp_t [**Del90**], and recently $\mathfrak{S}_t$ [**Del07**] for the rank $t \in \mathbb{C}$, which is not necessarily an integer. These are symmetric tensor categories which interpolate the ordinal representation categories similarly as described above. Comes and his coauthors study the structures of these categories for $\mathfrak{S}_t$ [**CO11, CK12**] and GL_t [**CW12, Com12**] in detail. Recall that the Tannaka–Krein duality allows us to reconstruct an algebraic group from its representation category along with its symmetric tensor structure. In this point of view, by the duality we can regard these tensor categories as generalized groups.

The variations of Deligne's category are studied in several ways. Knop [**Kno06, Kno07**] defined a wide generalization of Deligne's category, which includes ones for the finite general linear groups $GL_t(\mathbb{F}_q)$ and the wreath product $G^t \rtimes \mathfrak{S}_t$ for a finite group G. The author [**Mor12**] also studied the wreath product of algebras as taking the symmetric tensor product $\mathcal{S}ym_t(\mathcal{C})$ of a category $\mathcal{C}$. Etingof [**Eti14**] considered many non-compact representations such as degenerate affine Hecke algebras studied by Mathew [**Mat13**].

Our $\underline{H}_t$-$\mathcal{M}od$ is considered as a q-analogue of the Deligne's category for $\mathfrak{S}_t$. In fact, when the classical case $q = 1$, $\underline{H}_t$-$\mathcal{M}od$ contains Deligne's category as a full subcategory. When $\Bbbk$ is a field of characteristic zero, Deligne's category consisting of all finitely presented objects in $\underline{H}_t$-$\mathcal{M}od$. In contrast, in the modular case our $\underline{H}_t$-$\mathcal{M}od$ has more objects than Deligne's category.

1. Stable structures of the module category

1.1. Parabolic modules. A composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of $n \in \mathbb{N}$ is a finite tuple of natural numbers $\lambda_i \in \mathbb{N}$ such that $|\lambda| := \sum_i \lambda_i = n$. For such λ, let $H_\lambda \subset H_n$ denotes the corresponding *parabolic subalgebra*

$$H_\lambda := H_{\lambda_1} \otimes H_{\lambda_2} \otimes \cdots \otimes H_{\lambda_r} \hookrightarrow H_n.$$

Let $\Bbbk m_\lambda$ be the *trivial module* of H_λ spanned by m_λ on which each $T_w \in H_\lambda$ acts by a scalar $q^{\ell(w)}$. Its induced H_n-module

$$M_\lambda := H_n \otimes_{H_\lambda} \Bbbk m_\lambda$$

is called the *parabolic module*. For example, $\mathbb{1}_n := M_{(n)}$ is the trivial module of H_n and $M_{(1^n)} \simeq H_n$ is its left regular representation where $(1^n) := (1, 1, \dots, 1)$.

To represent elements of these parabolic modules and homomorphisms between them, we here introduce notions on tableaux. As usual, the *Young diagram* of a composition λ is defined by

$$Y(\lambda) := \{(i,j) \mid 1 \le i,\ 1 \le j \le \lambda_i\}.$$

A *row-semistandard tableau* of shape λ is a function $\mathsf{T}\colon Y(\lambda) \to \{1, 2, \dots\}$ which satisfies $\mathsf{T}(i,j) \le \mathsf{T}(i,j+1)$ for each pair of adjacent boxes $(i,j), (i,j+1) \in Y(\lambda)$, that is, entries in each row of T are weakly increasing. The *weight* of such tableau is a composition $\mu = (\mu_1, \mu_2, \dots)$ whose i-th component is $\mu_i := \#\mathsf{T}^{-1}(i)$. We denote by $\mathrm{Tab}_{\lambda;\mu}$ the set of row-semistandard tableaux of shape λ and weight μ. For example,

$$\mathrm{Tab}_{(2,3);(3,1,0,1)} = \left\{ \begin{ytableau} 1 & 1 \\ 1 & 2 & 4 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 1 & 1 & 4 \end{ytableau}, \begin{ytableau} 1 & 4 \\ 1 & 1 & 2 \end{ytableau}, \begin{ytableau} 2 & 4 \\ 1 & 1 & 1 \end{ytableau} \right\}.$$

A row-semistandard tableau is called a *row-standard tableau* if its weight is (1^n). We denote by $\mathrm{Tab}_\lambda := \mathrm{Tab}_{\lambda;(1^n)}$ the set of row-standard tableaux of shape λ.

The $\Bbbk$-module M_λ (resp. $\mathrm{Hom}_{H_n}(M_\mu, M_\lambda)$) has a basis parametrized by the set Tab_λ (resp. $\mathrm{Tab}_{\lambda;\mu}$) described as follows. First for a row-standard tableau T, let $d(\mathsf{T}) \in \mathfrak{S}_n$ be a permutation obtained by reading its entries from left to right for each rows from top to bottom. For example,

$$\mathsf{T} = \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 & 7 & 8 \\ 6 \end{ytableau} \quad \text{corresponds to} \quad d(\mathsf{T}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = s_3 s_4 s_6 s_7.$$

For each $\mathsf{T} \in \mathrm{Tab}_\lambda$, we let $m_\mathsf{T} := T_{d(w)} m_\lambda \in M_\lambda$. Then one can prove that the set $\{m_\mathsf{T} \mid \mathsf{T} \in \mathrm{Tab}_\lambda\}$ forms a basis of M_λ. Suppose each number i is contained in the $r(i)$-th row of T. The action of H_n on it is described as

$$T_i \cdot m_\mathsf{T} = \begin{cases} q m_\mathsf{T} & \text{if } r(i) = r(i+1), \\ m_{s_i \mathsf{T}} & \text{if } r(i) < r(i+1), \\ q m_\mathsf{T} + (q-1) m_{s_i \mathsf{T}} & \text{if } r(i) > r(i+1). \end{cases}$$

Next take two compositions λ, μ of n. For $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, we denote by Tab_S the set $\{\mathsf{T} \in \mathrm{Tab}_\lambda \mid \mathsf{T}|_\mu = \mathsf{S}\}$ where $\mathsf{T}|_\mu$ is a row-semistandard tableau of weight μ obtained from T by replacing its entries $1, 2, \ldots, \mu_1$ by 1, $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ by 2, and so forth. For example, for

$$\mathsf{S} = \begin{ytableau} 1 & 1 & 2 & 3 \\ 1 & 4 & 4 \\ 3 \end{ytableau},$$

we have

$$\mathrm{Tab}_\mathsf{S} = \left\{ \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 & 7 & 8 \\ 6 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 & 5 \\ 2 & 7 & 8 \\ 6 \end{ytableau}, \begin{ytableau} 2 & 3 & 4 & 5 \\ 1 & 7 & 8 \\ 6 \end{ytableau}, \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 & 7 & 8 \\ 5 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 & 6 \\ 2 & 7 & 8 \\ 5 \end{ytableau}, \begin{ytableau} 2 & 3 & 4 & 6 \\ 1 & 7 & 8 \\ 5 \end{ytableau} \right\}.$$

Since M_μ is a cyclic module, an H_n-homomorphism $M_\mu \to M_\lambda$ is determined by the value on m_μ. In this point of view, we have an isomorphism

$$\mathrm{Hom}_{H_n}(M_\mu, M_\lambda) \simeq \{x \in M_\lambda \mid T_w x = q^{\ell(w)} x \text{ for all } T_w \in H_\mu\}.$$

The right-hand side has a basis $\{m_\mathsf{S} \mid \mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}\}$ where $m_\mathsf{S} := \sum_{\mathsf{T} \in \mathrm{Tab}_\mathsf{S}} m_\mathsf{T}$. When there are no risks of confusions, we denote by the same symbol m_S the corresponding H_n-homomorphism $M_\mu \to M_\lambda$.

1.2. Induced modules. The direct sum category $\bigoplus_n (H_n\text{-}\mathcal{M}od)$ has the structure of tensor category by the convolution product $*$ defined as

$$V * W := H_{m+n} \otimes_{H_{(m,n)}} (V \boxtimes W) \in H_{m+n}\text{-}\mathcal{M}od$$

for $V \in H_m\text{-}\mathcal{M}od$ and $W \in H_n\text{-}\mathcal{M}od$, where $\boxtimes$ denotes the outer tensor product of modules. For example, for a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ the parabolic module M_λ can be expressed as

$$M_\lambda \simeq \mathbb{1}_{\lambda_1} * \mathbb{1}_{\lambda_2} * \cdots * \mathbb{1}_{\lambda_r}.$$

We define the *induction functor* as taking convolution with the trivial module. It plays a central role in what follows.

DEFINITION 1.1. Let $k, n \in \mathbb{N}$. For an H_n-module V, we denote by $\mathrm{Ind}_k V$ the H_{k+n}-module

$$\mathrm{Ind}_k V := \mathbb{1}_k * V.$$

This defines a functor $\mathrm{Ind}_k \colon H_n\text{-}\mathcal{M}od \to H_{k+n}\text{-}\mathcal{M}od$ between module categories.

Since the action $H_{k+n} \curvearrowleft H_{(k,n)}$ is free, it follows that the functor Ind_k is exact. More strongly we can prove that this functor has both left and right adjoint.

DEFINITION 1.2. Let $k, n \in \mathbb{N}$. For an H_{k+n}-module W, we define H_n-modules

$$\begin{aligned}
\mathrm{Res}_k W &:= \mathrm{Hom}_{H_{(k,n)}}(\mathbb{1}_k \boxtimes H_n, W|_{(k,n)}) \\
&\simeq \{x \in W \mid T_i x = qx \text{ for } 1 \le i \le k\}, \\
\mathrm{Res}'_k W &:= (H_n \boxtimes \mathbb{1}^*_k) \otimes_{H_{(n,k)}} W|_{(n,k)} \\
&\simeq W/{\sim}, \text{ where } T_i x \sim qx \text{ for } n+1 \le i \le n+k
\end{aligned}$$

where we denote by $W|_\lambda$ the restricted H_λ-module. Res_k and Res'_k are functors $H_{k+n}\text{-}\mathcal{M}od \to H_n\text{-}\mathcal{M}od$.

PROPOSITION 1.3. Res_k *(resp.* Res'_k*) is a right (resp. left) adjoint of* Ind_k*.*

Now for a convention, we let $H_n\text{-}\mathcal{M}od := \{0\}$ for $n \in \mathbb{Z}$ such that $n < 0$. We also extend the definitions of the functors above for $k, n \in \mathbb{Z}$ so that these are zero when the condition $k, n \ge 0$ is not satisfied. For two induced modules, we can describe the set of homomorphisms between them as follows.

THEOREM 1.4. *Let $d, m, n \in \mathbb{N}$ such that $m, n \le d$. For each $V \in H_m\text{-}\mathcal{M}od$ and $W \in H_n\text{-}\mathcal{M}od$, there is an isomorphism of $\Bbbk$-modules*

$$\mathrm{Hom}_{H_d}(\mathrm{Ind}_{d-m} V, \mathrm{Ind}_{d-n} W) \simeq \bigoplus_{m+n-d \le i} \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i} V, \mathrm{Res}_{n-i} W)$$

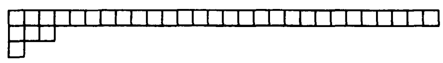
natural in V and W.

The summand in the right-hand side above is zero unless $0 \le i \le m, n$. In particular, as we varies the rank $d \in \mathbb{N}$ larger, this set is stable for $d \ge m+n$. This phenomenon suggests us that there should be a category which covers $H_d\text{-}\mathcal{M}od$ and has the hom-sets in the form

$$\bigoplus_i \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i} V, \mathrm{Res}_{n-i} W),$$

the stable set which does not depend on d. We realize this imaginary category as $\underline{H}_d\text{-}\mathcal{M}od$, the category of fakemodules, in the next section.

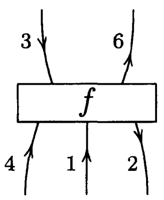
Note that a parabolic module M_λ is a special case of induced modules, that is, we can write $M_\lambda \simeq \mathrm{Ind}_{\lambda_1} M_{\lambda'}$ where $\lambda' = (\lambda_2, \dots, \lambda_r)$. Taking $d \gg 0$ corresponds to considering Young diagrams whose first rows are very long:

.

The theorem says that the structure of the module category will be stable for such Young diagrams with sufficiently long first rows.

1.3. String diagrams. In order to explain the isomorphism in Theorem 1.4 precisely, we introduce *string diagrams* which are useful for calculation in theory of 2-categories. In a diagram we visualize a functor by a colored string. The right (resp. left) region separated by a string stands for the domain (resp. codomain) category of the corresponding functor. A composite of these functors is represented by a sequence of strings arranged horizontally. In particular, the identity functor is represented by the "no strings" diagram. A natural transformation between such functors are represented by a figure connecting these sequences from top to bottom.

In this report, we represent the functor Ind_k by a down arrow $\downarrow$, and both Res_k, Res'_k by up arrows $\uparrow$ which are labeled by k. For example, $f\colon \mathrm{Ind}_3\mathrm{Res}_6 \to \mathrm{Res}_4\mathrm{Res}_1\mathrm{Ind}_2$ is represented by a figure like

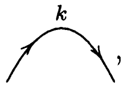


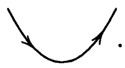
Note that the diagram above can not distinguish Res_k from Res'_k, but we only use diagrams when it is clear from the context.

The adjointness between Ind_k and Res_k yields natural transformations

$$\delta_k\colon \mathrm{Id} \to \mathrm{Res}_k\mathrm{Ind}_k, \qquad \epsilon_k\colon \mathrm{Ind}_k\mathrm{Res}_k \to \mathrm{Id}$$

called the unit and the counit respectively. We represent these morphisms by the cap and the cup diagrams:

$\delta_k =$ 

, $\epsilon_k =$ 

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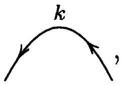
We also have the the unit $\delta'_k\colon \mathrm{Id} \to \mathrm{Ind}_k\mathrm{Res}'_k$ counit $\epsilon'_k\colon \mathrm{Res}'_k\mathrm{Ind}_k \to \mathrm{Id}$ induced by the other adjunction. We represent them by the same diagrams as above but arrows are reversed:

$\delta'_k =$ 

, $\epsilon'_k =$ 

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Now let $k, l \in \mathbb{N}$. We define three H_{k+l}-homomorphisms

$$\mu_{(k,l)}\colon M_{(k,l)} \to \mathbb{1}_{k+l}, \qquad \Delta_{(k,l)}\colon \mathbb{1}_{k+l} \to M_{(k,l)} \qquad \sigma_{(k,l)}\colon M_{(l,k)} \to M_{(k,l)}$$

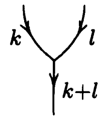
which correspond to the row-semistandard tableaux

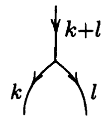
$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&2&2&\cdots&2\\\hline\end{array}\,, \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}\,, \qquad \begin{array}{|c|c|c|c|c|c|}\hline 2&2&\cdots&2&2&2\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}$$

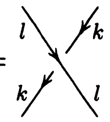
respectively. These homomorphisms induce natural transformations between functors $H_n\text{-}\mathcal{M}od \to H_{k+l+n}\text{-}\mathcal{M}od$,

$$\mu_{(k,l)}\colon \mathrm{Ind}_k\mathrm{Ind}_l \to \mathrm{Ind}_{k+l}, \quad \Delta_{(k,l)}\colon \mathrm{Ind}_{k+l} \to \mathrm{Ind}_k\mathrm{Ind}_l, \quad \sigma_{(k,l)}\colon \mathrm{Ind}_l\mathrm{Ind}_k \to \mathrm{Ind}_k\mathrm{Ind}_l$$

which we denote by the same symbols. Again, if k and l do not satisfy $k, l \geq 0$, then these morphisms are defined to be zero. We represent these natural transformations by the string diagrams

$\mu_{(k,l)} =$ 

, $\Delta_{(k,l)} =$ 

, $\sigma_{(k,l)} =$ 

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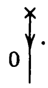
that is, junction, branch, and crossing of strings respectively. Finally, an obvious isomorphism $\mathrm{Ind}_0 \simeq \mathrm{Id}$ and its inverse are represented by broken strings:

and 

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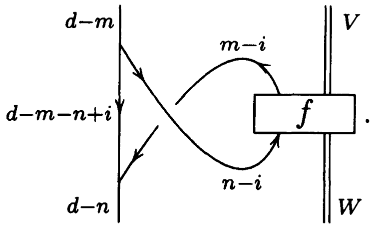
Now the isomorphism in Theorem 1.4 can be represented as follows. For an H_i-homomorphism $f\colon \mathrm{Res}'_{m-i}V \to \mathrm{Res}_{n-i}W$ in the right-hand side, the corresponding H_d-homomorphism $\mathrm{Ind}_{d-m}V \to \mathrm{Ind}_{d-n}W$ can be illustrated as



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The theorem follows by studying the double cosets $\mathfrak{S}_{(d-m,m)} \curvearrowright \mathfrak{S}_n \curvearrowleft \mathfrak{S}_{(d-n,n)}$.

2. The category of fakemodules

2.1. Binomial sequences. Before we proceed to the main definition, we need to explain the index set $B_q(\Bbbk)$ of q-binomial sequences. The key point of our theory is that the composition of certain H_n-homomorphism can be computed by *q-binomial coeffients.* Here a q-binomial coeffient $\begin{bmatrix} n \\ k \end{bmatrix}$ for $n, k \in \mathbb{N}$ is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n][n-1]\cdots[n-k+1]}{[1][2]\cdots[k]} = \frac{[n]!}{[k]![n-k]!}$$

where $[i] := 1+q+\cdots+q^{i-1}$ is the q-integer and $[i]! := [1][2]\cdots[i]$ is the q-factorial. This rational function for the variable q actually is in the polynomial ring $\mathbb{Z}[q]$, so assigning the actual value of $q \in \Bbbk$ we can regard $\begin{bmatrix} n \\ k \end{bmatrix} \in \Bbbk$. A q-binomial sequence is a generalization of the function $\mathbb{N} \to \Bbbk$; $k \mapsto \begin{bmatrix} n \\ k \end{bmatrix}$ for $n \in \mathbb{N}$.

DEFINITION 2.1. A *q-binomial sequence* in $\Bbbk$ is a function $t\colon \mathbb{N} \to \Bbbk$, whose values are written as $k \mapsto \begin{bmatrix} t \\ k \end{bmatrix}$, which satisfies

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} t \\ k \end{bmatrix}\begin{bmatrix} t \\ l \end{bmatrix} = \sum_{0 \le i \le k,l} q^{(k-i)(l-i)} \begin{bmatrix} l \\ i \end{bmatrix}\begin{bmatrix} k+l-i \\ l \end{bmatrix}\begin{bmatrix} t \\ k+l-i \end{bmatrix}.$$

We denote by $B_q(\Bbbk)$ the set of all q-binomial sequences in $\Bbbk$.

When $q = 1$, a 1-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is just the ordinary binomial coefficient $\binom{n}{k}$. So we prefer writing a value of 1-binomial sequence as $\binom{t}{k}$ rather than $\begin{bmatrix} t \\ k \end{bmatrix}$.

As described above, each natural number defines a q-binomial sequence. Namely:

LEMMA 2.2. *For each $n \in \mathbb{N}$, the function $k \mapsto \begin{bmatrix} n \\ k \end{bmatrix}$ is a q-binomial sequence. Moreover, the map $\mathbb{N} \to B_q(\Bbbk)$ is injective.*

Henceforth we regard $\mathbb{N}$ as a subset of $B_q(\Bbbk)$ via this embedding. Interestingly the addition on $\mathbb{N}$ can be lifted to the whole set $B_q(\Bbbk)$ as follows, which makes it into a commutative monoid.

PROPOSITION 2.3. *For two q-binomial sequences t and u, let*

$$\begin{bmatrix} t+u \\ k \end{bmatrix} := \sum_{0\le i\le k} q^{\binom{i}{2}}(q-1)^i[i]! \sum_{0\le j\le k-i} \begin{bmatrix} k-j \\ i \end{bmatrix}\begin{bmatrix} i+j \\ i \end{bmatrix}\begin{bmatrix} t \\ k-j \end{bmatrix}\begin{bmatrix} u \\ i+j \end{bmatrix}.$$

Then $t+u$ is also a q-binomial sequence. $B_q(\Bbbk)$ forms a commutative monoid with respect to this addition and the unit element 0.

Now in the examples below we assume that $\Bbbk$ is a field. We here give the complete classification of q-binomial sequences under this assumption. Let $e := \operatorname{char}_q \Bbbk$ be the *q-characteristic of* $\Bbbk$, the minimum positive number such that $[e] = 0$. When there are no such numbers we let $e = 0$ for convention.

EXAMPLE 2.4. When $q = 0$, we have $B_0(\Bbbk) = \mathbb{N} \cup \{\infty\}$. Here ∞ is a q-binomial sequence which satisfy $\infty + t = \infty$ for all $t \in B_q(\Bbbk)$ defined by

$$\begin{bmatrix} \infty \\ k \end{bmatrix} := \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

when $1 - q^i$ $(i \ge 1)$ are all invertible. In this case we have just $\left[\begin{smallmatrix} \infty \\ k \end{smallmatrix}\right] = 1$ for all k.

EXAMPLE 2.5. Suppose $q \ne 0$ and $e = 0$. Then the map

$$\begin{aligned} B_q(\Bbbk) &\to \Bbbk \\ t &\mapsto [t] \end{aligned}$$

where $[t] := \left[\begin{smallmatrix} t \\ 1 \end{smallmatrix}\right]$ is bijective. For each $x \in \Bbbk$, the corresponding $t \in B_q(\Bbbk)$ such that $x = [t]$ is given by

$$\begin{bmatrix} t \\ k \end{bmatrix} := q^{-\binom{i}{2}} \frac{x(x-[1])\cdots(x-[k-1])}{[1][2]\cdots[k]}.$$

Beware that this bijection does not preserve addition unless $q = 1$.

EXAMPLE 2.6. When $e > 0$, there is an exact sequence of commutative monoids

$$0 \to B_1(\Bbbk) \to B_q(\Bbbk) \to \mathbb{Z}/e\mathbb{Z} \to 0.$$

Moreover if $q = 1$ (i.e. $e = \operatorname{char}\Bbbk > 0$) naturally $B_1(\Bbbk) = \mathbb{Z}_e$, the set of e-adic integers. In this case the values for $n \in \mathbb{Z}_e$ is given by

$$\binom{n}{k} := \binom{n \bmod e^k}{k}$$

which is well-defined modulo e by Lucas's theorem.

We will use such a q-binomial sequence t to specify the "rank" of the "Iwahori–Hecke algebra $\underline{H}_t$". However, in the following construction of its fakemodule category, we will need to use q-binomial sequencs "$t - m$" for all $m \in \mathbb{N}$. Its uniqueness is guaranteed by the next lemma, while its existence is not in general.

LEMMA 2.7. *The shift map $B_q(\Bbbk) \to B_q(\Bbbk)$; $t \mapsto t+1$ is injective. It is also surjective if and only if $q \in \Bbbk$ is invertible.*

Hence we have to use q-binomial sequences only which have following property:

Definition 2.8. A *q-binomial sequence* t in $\Bbbk$ is said to be *total* if $t-m$ exists for all $m \in \mathbb{N}$. We denote by $B_q^+(\Bbbk)$ the set of total q-binomial sequences; so

$$B_q^+(\Bbbk) := \bigcap_{m\in\mathbb{N}} (B_q(\Bbbk)+m).$$

The subset $B_q^+(\Bbbk)$ is an ideal of $B_q(\Bbbk)$ with respect to the addition. As we noted above, if q is invertible then $B_q^+(\Bbbk) = B_q(\Bbbk)$. On the other hand, for $q=0$ we have just $B_0^+(\Bbbk) = \{\infty\}$.

2.2. The category of induced fakemodules. Now let $t \in B_q^+(\Bbbk)$ be a total q-binomial sequence in $\Bbbk$. In order to define the whole category $\underline{H_t}\text{-}\mathcal{M}od$, we first need to introduce its full subcategory $\underline{H_t}\text{-}\mathcal{M}od_0$. An object of $\underline{H_t}\text{-}\mathcal{M}od_0$ is written as $\underline{\mathrm{Ind}}_{t-m}V$, and called an *induced fakemodule*. It is made to imitate the behaviors of the ordinary induced module $\mathrm{Ind}_{d-m}V$ which we introduced in the previous section. We define the category $\underline{H_t}\text{-}\mathcal{M}od_0$ in terms of generators and relations as follows.

Definition 2.9. An object in the category $\underline{H_t}\text{-}\mathcal{M}od_0$ is an H_m-module V for some $m \in \mathbb{N}$, represented by the symbol $\underline{\mathrm{Ind}}_{t-m}V$. Morphisms between these objects are generated by

$$\underline{\mathrm{Ind}}_{t-m}f\colon \underline{\mathrm{Ind}}_{t-m}V \to \underline{\mathrm{Ind}}_{t-m}W,$$

defined for each H_m-homomorphism $f\colon V\to W$, and

$$\underline{\mu}_{(t-m-k,k)}V\colon \underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V \to \underline{\mathrm{Ind}}_{t-m}V,$$
$$\underline{\Delta}_{(t-m-k,k)}V\colon \underline{\mathrm{Ind}}_{t-m}V \to \underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V$$

defined for each H_m-module V and $k\in\mathbb{N}$, with relations listed below. The first two of them are:

(a) $\underline{\mathrm{Ind}}_{t-m}$ is a $\Bbbk$-linear functor $H_m\text{-}\mathcal{M}od \to \underline{H_t}\text{-}\mathcal{M}od$. That is,

$$\underline{\mathrm{Ind}}_{t-m}\mathrm{id}_V = \mathrm{id}_{\underline{\mathrm{Ind}}_{t-m}V},\quad \underline{\mathrm{Ind}}_{t-m}(f\circ g) = \underline{\mathrm{Ind}}_{t-m}f \circ \underline{\mathrm{Ind}}_{t-m}g$$

and

$$\underline{\mathrm{Ind}}_{t-m}(af+bg) = a\cdot \underline{\mathrm{Ind}}_{t-m}f + b\cdot \underline{\mathrm{Ind}}_{t-m}g$$

for suitable H_m-homomorphisms f,g and scalars $a,b\in\Bbbk$.

(b) $\underline{\mu}_{(t-m-k,k)}$ and $\underline{\Delta}_{(t-m-k,k)}$ are both natural transformations between functors $H_m\text{-}\mathcal{M}od \to \underline{H_t}\text{-}\mathcal{M}od$, respectively $\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k \rightleftharpoons \underline{\mathrm{Ind}}_{t-m}$. That is, the square below and its dual commute for any H_m-homomorphism $f\colon V\to W$:

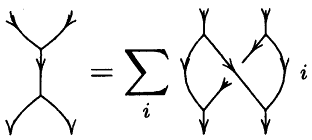
$$\begin{tikzcd}
\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V \arrow[r, "\underline{\mu}_{(t-m-k,k)}V"] \arrow[d, "\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k f"'] & \underline{\mathrm{Ind}}_{t-m}V \arrow[d, "\underline{\mathrm{Ind}}_{t-m}f"] \\
\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k W \arrow[r, "\underline{\mu}_{(t-m-k,k)}W"'] & \underline{\mathrm{Ind}}_{t-m}W.
\end{tikzcd}$$

The rest relations are represented by diagrams as we do before. To represent the functor $\underline{\mathrm{Ind}}$ and the natural transformations $\underline{\mu}$ and $\underline{\Delta}$, we use same diagrams as Ind, μ and Δ. Here arrows which represent $\underline{\mathrm{Ind}}$ always appear in leftmost of each diagram.

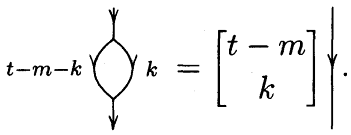
(1) The associativity and the coassociativity laws:

(2) The unit and the counit laws:

(3) The graded bialgebra relation:



(4) The bubble elimination:



As we mentioned above, an object and a morphism in $\underline{H}_t$-$\mathcal{M}od$ is called an $\underline{H}_t$-*fakemodule* and an $\underline{H}_t$-*fakemorphism* respectively. We denote by $\mathrm{Hom}_{\underline{H}_t}$ the set of fakemorphisms between fakemodules instead of $\mathrm{Hom}_{\underline{H}_t\text{-}\mathcal{M}od_0}$ for simplicity.

When the rank t is an usual integral rank $d \in \mathbb{N}$, the relations above are easily verified. Hence there is a canonical functor $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ which sends $\underline{\mathrm{Ind}}_{d-m}$ to Ind_{d-m}, $\underline{\mu}_{(d-m-k,k)}$ to $\mu_{(d-m-k,k)}$ and $\underline{\Delta}_{(d-m-k,k)}$ to $\Delta_{(d-m-k,k)}$. We call it the *realization functor* which realize a module from a fakemodule. Note that we have a natural isomorphism $\mathrm{Ind}_0 V \simeq V$, and $\mathrm{Ind}_0 f = f$ for $f\colon V \to W$ via this isomorphism. The category $\underline{H}_d$-$\mathcal{M}od_0$ has the corresponding object $\underline{\mathrm{Ind}}_0 V$ and the morphism $\underline{\mathrm{Ind}}_0 f$ so that the functor P is full and surjective. However, in the definition we use values of q-binomial coefficients $\left[\begin{smallmatrix} d-m \\ k \end{smallmatrix}\right]$ for negative integers. So in order to define $\underline{H}_d$-$\mathcal{M}od_0$ we must have that the q-binomial sequence d is total, or equivalently, $q \in \Bbbk$ is invertible. Summarizing the above:

PROPOSITION 2.10. *Suppose that $q \in \Bbbk$ is invertible. Then for each $d \in \mathbb{N}$, there is a full and surjective functor $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ such that $P\,\underline{\mathrm{Ind}}_{d-m} = \mathrm{Ind}_{d-m}$, $P\underline{\mu}_{(d-m-k,k)} = \mu_{(d-m-k,k)}$ and $P\underline{\Delta}_{(d-m-k,k)} = \Delta_{(d-m-k,k)}$.*

We remark that if $m > d$ then a module $\mathrm{Ind}_{d-m}V$ is zero by definition while the corresponding fakemodule $\underline{\mathrm{Ind}}_{d-m}V$ is not. Actually, the kernel of the realization P is generated by such fakemodules.

As we claimed before, we can completely describe the set of fakemorphisms in $\underline{H}_t$-$\mathcal{M}od_0$ as follows.

THEOREM 2.11. *For $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$, we have*

$$\mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) \simeq \bigoplus_i \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).$$

This isomorphism is defined similarly as in Theorem 1.4 using $\underline{\Delta}$ and $\underline{\mu}$ instead of Δ and μ according to the string diagram in §1.3.

From this result immediately we obtain the statement below.

COROLLARY 2.12. *Suppose q is invertible and let $d \in \mathbb{N}$. For $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$, the realization on the morphisms*

$$\mathrm{Hom}_{\underline{H}_d}(\underline{\mathrm{Ind}}_{d-m}V, \underline{\mathrm{Ind}}_{d-n}W) \twoheadrightarrow \mathrm{Hom}_{H_d}(\mathrm{Ind}_{d-m}V, \mathrm{Ind}_{d-n}W)$$

is an isomorphism when $d \geq m+n$.

The fakemodule category has more morphisms than the ordinal module category, which is usually hidden from our view.

2.3. Parabolic fakemodules. Recall that induction is taking convolution product with the trivial module. By the definition of the category, we can define convolution product of a fakemodule and a usual module as

$$(\underline{\mathrm{Ind}}_{t-m}V) * W := \underline{\mathrm{Ind}}_{t-m}(V * W)$$

for each $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$. It defines a functor

$$*\colon \underline{H}_t\text{-}\mathcal{M}od_0 \times H_n\text{-}\mathcal{M}od \to \underline{H}_{t+n}\text{-}\mathcal{M}od_0.$$

We denote by $\underline{\mathbb{1}}_t$ the *trivial fakemodule* $\underline{\mathrm{Ind}}_t \mathbb{1}_0$. Then an induced fakemodule can be also written as $\underline{\mathrm{Ind}}_{t-m}V \simeq \underline{\mathbb{1}}_{t-m} * V$ using the convolution. This product is also associative, so it provides a structure of right $\bigoplus_n(H_n$-$\mathcal{M}od)$-module for the category $\bigoplus_m(\underline{H}_{t+m}$-$\mathcal{M}od_0)$.

Recall again that a parabolic module M_λ is a special case of an induced module. We here introduce parabolic fakemodules into our category $\underline{H}_t$-$\mathcal{M}od_0$ by imitating this construction.

DEFINITION 2.13. Let t be a total q-binomial sequence. A *fakecomposition* $\lambda = (\lambda_1, \lambda')$ of t is a pair of a total q-binomial sequence λ_1 and a composition λ' such that $|\lambda| := \lambda_1 + |\lambda'| = t$. For such λ, we write $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ where $\lambda_i := \lambda'_{i-1}$ for $i \geq 2$. Let $\underline{M}_\lambda \in \underline{H}_t$-$\mathcal{M}od_0$ be a fakemodule defined by

$$\underline{M}_\lambda := \underline{\mathrm{Ind}}_{\lambda_1} M_{\lambda'} \simeq \underline{\mathbb{1}}_{\lambda_1} * \mathbb{1}_{\lambda_2} * \mathbb{1}_{\lambda_3} * \cdots * \mathbb{1}_{\lambda_l}.$$

Let λ and μ be two fakecompositions of t. Let $\lambda|_d$ and $\mu|_d$ be corresponding fakecompositions of $d \in \mathbb{N}$ obtained by replacing their first components. By Theorem 2.11 the set of H_d-homomorphisms $M_{\mu|d} \to M_{\lambda|d}$ stabilizes for sufficiently large d into the set of $\underline{H}_t$-fakemorphisms $\underline{M}_\mu \to \underline{M}_\lambda$. So as a basis of $\mathrm{Hom}_{\underline{H}_d}(\underline{M}_\mu, \underline{M}_\lambda)$ we can take the set $\mathrm{Tab}_{\lambda|d;\mu|d}$ for $d \gg 0$ which converges to a finite set. Formally we define

$$\underline{\mathrm{Tab}}_{\lambda;\mu} := \varinjlim_d \mathrm{Tab}_{\lambda|d;\mu|d}$$

where the map $\mathrm{Tab}_{\lambda|d;\mu|d} \hookrightarrow \mathrm{Tab}_{\lambda|d+1;\mu|d+1}$ is inserting $\boxed{1}$ on the first row of the tableau from left. For example, when $\lambda = (t-2, 2)$ and $\mu = (t-3, 2, 1)$, the set $\underline{\mathrm{Tab}}_{(t-2,2);(t-3,2,1)}$ consisting of the tableaux

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&3\\\hline 2&2\\\cline{1-2}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&2&2\\\hline 1&3\\\cline{1-2}\end{array}\ ,$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&2&3\\\hline 1&2\\\cline{1-2}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&2&3\\\hline 1&1\\\cline{1-2}\end{array}$$

regardless of t. We denote by the symbol $\underline{m}_{\mathsf{S}}$ the fakemorphism $\underline{M}_{\mu} \to \underline{M}_{\lambda}$ corresponding to S, so that the set $\{\underline{m}_{\mathsf{S}} \mid \mathsf{S} \in \underline{\mathrm{Tab}}_{\lambda;\mu}\}$ is a basis of $\mathrm{Hom}_{\underline{H}_t}(\underline{M}_{\mu}, \underline{M}_{\lambda})$. We can also compute the composition of such fakemorphisms by regarding t as a large number.

When q is invertible, for a fakecomposition λ of $d \in \mathbb{N}$ the realization functor P sends the fakemodule $\underline{M}_{\lambda}$ to M_{λ} if λ is a composition (that is, $\lambda_1 \geq 0$) and otherwise 0. For two compositions λ and μ, the realization of fakemorphisms is given by

$$P\colon \mathrm{Hom}_{\underline{H}_d}(\underline{M}_{\mu}, \underline{M}_{\lambda}) \twoheadrightarrow \mathrm{Hom}_{H_d}(M_{\mu}, M_{\lambda})$$

$$\underline{m}_{\mathsf{S}} \mapsto \begin{cases} m_{\mathsf{S}} & \text{if } \mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}, \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, to realize the $\underline{H}_d$-fakemorphism $\underline{m}_{\mathsf{S}}$ to an H_d-homomorphism m_{S}, we should cut off superfluous $\boxed{1}$'s in the first row of S. When there are not enough such $\boxed{1}$'s, it produces a zero homomorphism. If $t = 4$ in the example above, the realization map for $\lambda = (2,2)$ and $\mu = (1,2,1)$ is given by

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 1 & 2 \\ 2 & 3 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau}, \qquad \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 1 & 3 \\ 2 & 2 \end{ytableau} \mapsto \begin{ytableau} 1 & 3 \\ 2 & 2 \end{ytableau},$$

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 2 & 2 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} 2 & 2 \\ 1 & 3 \end{ytableau}, \qquad \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 2 & 3 \\ 1 & 2 \end{ytableau} \mapsto \begin{ytableau} 2 & 3 \\ 1 & 2 \end{ytableau},$$

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 2 & 2 & 3 \\ 1 & 1 \end{ytableau} \mapsto 0$$

where we represent morphisms m_{S} and $\underline{m}_{\mathsf{S}}$ by a tableau S itself for short.

2.4. Completion of category. Unfortunately, the category $\underline{H}_t$-$\mathcal{M}od_0$ lacks the ability to apply various categorical operations. We here see that the category $\underline{H}_t$-$\mathcal{M}od_0$ can be naturally embedded to a larger category $\underline{H}_t$-$\mathcal{M}od$ which admits several operations. The category $\underline{H}_t$-$\mathcal{M}od$ is constructed from $\underline{H}_t$-$\mathcal{M}od_0$ using the process of two completions of category, namely *pseudo-abelian envelope* (see [**Del07**, §1]) and *indization* (see [**KS06**, §6]).

DEFINITION 2.14. Let $\underline{H}_t$-mod_0 be the full subcategory of $\underline{H}_t$-$\mathcal{M}od_0$ consisting of objects $\underline{\mathrm{Ind}}_{t-m}V$ such that V is finitely presented. Then we put

$$\underline{H}_t\text{-}mod := (\underline{H}_t\text{-}mod_0)^{\mathrm{psab}},$$

the *pseudo-abelian envelope* of the category $\underline{H}_t$-mod_0. That is, an object in $\underline{H}_t$-mod is a direct summand of a formal direct sum of objects in $\underline{H}_t$-mod_0.

$\underline{H}_t$-mod_0 is considered as the "category of finitely presented $\underline{H}_t$-fakemodules". Note that it contains all parabolic fakemodules $\underline{M}_{\lambda}$. Recall that for an algebra A, any A-module is a direct limit (i.e. filtered colimit) of finitely presented ones. Based on this observation, we introduce the definition of the whole fakemodule category $\underline{H}_t$-$\mathcal{M}od$ as follows.

DEFINITION 2.15. Let

$$\underline{H}_t\text{-}\mathcal{M}od := (\underline{H}_t\text{-}mod)^{\mathrm{ind}}$$

be the *indization* of the category $\underline{H}_t$-mod. That is, an object in $\underline{H}_t$-$\mathcal{M}od$ is a formal direct limit (ind-object) of objects in $\underline{H}_t$-mod.

Now it follows by definition.

PROPOSITION 2.16. *The category $\underline{H}_t$-$\mathcal{M}od$ is closed under taking direct sums, direct summands and direct limits. $\underline{H}_t$-mod is a full subcategory of $\underline{H}_t$-$\mathcal{M}od$ consisting of finitely presented (or compact) objects.*

We define the embedding functor $\underline{H}_t\text{-}\mathcal{M}od_0 \to \underline{H}_t\text{-}\mathcal{M}od$ as follows. Recall that an object in $\underline{H}_t\text{-}\mathcal{M}od_0$ is the induced fakemodule $\underline{\mathrm{Ind}}_{t-m}V$ of an arbitrary H_m-module V. We can represent V as a direct limit of finitely presented modules $V \simeq \varinjlim_i V_i$. Via the embedding, the object $\underline{\mathrm{Ind}}_{t-m}V \in \underline{H}_t\text{-}\mathcal{M}od_0$ is mapped to the direct limit $\varinjlim_i(\underline{\mathrm{Ind}}_{t-m}V_i) \in \underline{H}_t\text{-}\mathcal{M}od$ of finitely presented fakemodules. Then one can prove the following.

PROPOSITION 2.17. *The functor $\underline{H}_t\text{-}\mathcal{M}od_0 \to \underline{H}_t\text{-}\mathcal{M}od$ is well-defined and fully faithful.*

We still have the realization $P\colon \underline{H}_d\text{-}\mathcal{M}od \to H_d\text{-}\mathcal{M}od$ for $d \in \mathbb{N}$, and similarly several functors $\underline{H}_t\text{-}\mathcal{M}od_0 \to \mathcal{C}$ can be extended to $\underline{H}_t\text{-}\mathcal{M}od \to \mathcal{C}$.

2.5. Comparison with Deligne's category. Now assume the classical case $q = 1$, so in particular every 1-binomial sequence is total. Since in this case we have an isomorphism $H_n \simeq \Bbbk\mathfrak{S}_n$, it seems better to denote our category by $\Bbbk\underline{\mathfrak{S}}_t\text{-}\mathcal{M}od$ rather than $\underline{H}_t\text{-}\mathcal{M}od$. Recall that $\Bbbk\mathfrak{S}_n\text{-}\mathcal{M}od$ has a tensor product of modules over $\Bbbk$, defined through the diagonal embedding $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_n \times \mathfrak{S}_n$. We can lift this tensor product on the fakemodule category $\Bbbk\underline{\mathfrak{S}}_t\text{-}\mathcal{M}od$.

THEOREM 2.18. *$\Bbbk\underline{\mathfrak{S}}_t\text{-}\mathcal{M}od$ has a canonical structure of tensor category such that for each $d \in \mathbb{N}$ the realization*

$$P\colon \Bbbk\underline{\mathfrak{S}}_d\text{-}\mathcal{M}od \to \Bbbk\mathfrak{S}_d\text{-}\mathcal{M}od$$

is a tensor functor.

We finish this report by describing the relation between the motivating Deligne's category [**Del07**] and $\Bbbk\underline{\mathfrak{S}}_t\text{-}\mathcal{M}od$. For $t \in B_1(\Bbbk)$, let $\mathcal{R}ep(\mathfrak{S}_t)$ denotes the Deligne's category for the rank $\binom{t}{1} \in \Bbbk$. It has an object $[m] \in \mathcal{R}ep(\mathfrak{S}_t)$ for each $m \in \mathbb{N}$ which correspond to the parabolic fakemodule $\underline{M}_{(t-m,1^m)}$ in our notation, and $\mathcal{R}ep(\mathfrak{S}_t)$ is generated by these objects. We can define the functor

$$\begin{aligned} \mathcal{R}ep(\mathfrak{S}_t) &\to \Bbbk\underline{\mathfrak{S}}_t\text{-}mod \\ [m] &\mapsto \underline{M}_{(t-m,1^m)}. \end{aligned}$$

which is fully faithful and preserves tensor product. Hence we can regard that:

PROPOSITION 2.19. *Deligne's category $\mathcal{R}ep(\mathfrak{S}_t)$ is a tensor full subcategory of $\Bbbk\underline{\mathfrak{S}}_t$-mod.*

It is well-known that when $\Bbbk$ is a field of characteristic zero, the category $\mathcal{R}ep(\mathfrak{S}_m)$ is semisimple. Since every its simple object is obtained as a direct summand of the regular representation $M_{(1^m)} \simeq \Bbbk\mathfrak{S}_m$, we have a category equivalence $\mathcal{R}ep(\mathfrak{S}_t) \simeq \Bbbk\underline{\mathfrak{S}}_t\text{-}mod$. In contrast, if $\Bbbk$ has a positive characteristic then the image of the embedding $\mathcal{R}ep(\mathfrak{S}_t) \hookrightarrow \Bbbk\underline{\mathfrak{S}}_t\text{-}mod$ is a proper full subcategory. We remark that for each natural number $d \in \mathbb{N}$, Deligne's category $\mathcal{R}ep(\mathfrak{S}_d)$ only depends on the scalar value $d \in \Bbbk$ while our $\Bbbk\underline{\mathfrak{S}_d}\text{-}\mathcal{M}od$ gives different categories for each $d \in \mathbb{N}$. So $\Bbbk\underline{\mathfrak{S}_d}\text{-}\mathcal{M}od$ is considered to be capturing more precise structures in the modular case.

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